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It was suggested earlier that the gravitational redshift formula can be invalid when the effect of the backscattering is strong. It is demonstrated here numerically, for an exemplary electromagnetic pulse that is: i) initially located very close to the horizon of a Schwarzschild black hole and ii) strongly backscattered, that a mean frequency does not obey the standard redshift formula. Redshifts appear to depend on the frequency and there manifests a backscatter-induced blueshift in the outgoing radiation.

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I. INTRODUCTION

Standard derivations of the gravitational redshift base on the approximation of geometric optics (see, for a discussion, [1] - [4]). This problem has been recently reconsidered in the wave formulation [5]. It was shown, for a class of compact shocks that are separated in some sense [6] from the horizon of a Schwarzschild black hole, that the energy flux scales accordingly to the redshift formula. Introducing the concept of photons and assuming that their number is conserved, one again arrives at the standard relation for the frequency. As an alternative approach one can apply the Fourier analysis, as outlined below in Sec. II.

The crucial feature behind the above mentioned compactness and separateness conditions is that when they hold true, then the backscatter is negligible [8]. It was remarked in [5] that a compact pulse of radiation that is exposed to a significant backscattering must not obey the familiar redshift relation. This is suggested by the following reasoning. A spatially compact wave packet consists of a superposition of monochromatic waves of different frequencies. The effect of the backscattering is stronger for low frequency waves than for high frequency ones, that is spectral amplitudes of the former are stronger damped than those of the latter. Therefore one expects that the backscattering induces a shift in a mean frequency, so that it does not satisfy the gravitational redshift formula. The aim of this investigation is to extend results of [5] and find a numerical example in favour of this conjecture. This examplary wave pulse **must break** – and in fact **does so**, in accordance with [5] – the compactness condition of [6].

In the next section we write equations and review relevant results of the preceding papers [5], [7]. Sec. III describes an electromagnetic wave that does not comply to the standard redshift formula. Sec. IV presents a short summary.

II. GRAVITATIONAL REDSHIFT: CLASSICAL DERIVATION

The space-time geometry is defined by the Schwarzschild line element,

$$ds^2 = -(1-\frac{2m}{R})dt^2 + \frac{1}{1-\frac{2m}{R}}dR^2 + R^2 d\Omega^2 \ , \qquad (1)$$

where t is a time coordinate, R is an areal radius and $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ (with $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$) is the line element on the unit sphere. The Newtonian gravitational constant G and the velocity of light c are put equal to 1. We define the tortoise coordinate $r^* = R + 2m\ln(\frac{R}{2m} - 1)$ and $\eta_R \equiv 1 - \frac{2m}{R}$. We will consider only the dipole electromagnetic term and, more specifically, choose the potential one-form $A = \sqrt{3/2}\sin^2\theta\Psi(r^*, t)d\phi$. The unknown function Ψ satisfies the equation [9]

$$(-\partial_t^2 + \partial_{r^*}^2)\Psi = \eta_R \frac{2}{R^2}\Psi. \qquad (2)$$

Let the electromagnetic strength field tensor be $F_{\mu\nu}$. The stress-energy tensor of the electromagnetic field reads $T_\mu^\nu = F_{\mu\gamma}F^{\nu\gamma} - (1/4)g_\mu^\nu F_{\gamma\delta}F^{\gamma\delta}$ and the time-like translational Killing vector is denoted as ζ. One can define the energy flux $\hat{P}_R$,

$$\hat{P}_R(R,t) \equiv \frac{1}{\sqrt{\eta_r}} \int_{S(R)} dS(R) n_r T^{r\mu}\zeta_\mu. \qquad (3)$$

Here n is the unit normal to the sphere R and dS is the standard area element on S. $\hat{P}_R$ is equal to

$$\hat{P}_R(R,t) = -\frac{4\pi}{\sqrt{\eta_R}}\partial_t\Psi\partial_{r^*}\Psi. \qquad (4)$$

Let $\tilde{\Gamma}_{R_0}$ be a null geodesic directed outward from the point R_0 of the initial hypersurface and let $\tilde{\Gamma}_{R_0,(R,t)}$ be a segment of $\tilde{\Gamma}_{R_0}$ that connects R_0 and (R,t). Comparing the energy fluxes through the spheres $S(R)$ (where $R >> 2m$) and the initial $S(R_0)$, one obtains [5]

$$\hat{P}_R(R) \approx \sqrt{\frac{\eta_{R_0}}{\eta_R}}\hat{P}_R(R_0), \tag{5}$$

assuming that the initial support of a dipole wave packet is contained in the annulus (a,b) such that $(b-a)/(a\eta_a^5) << 1$. Here a and b are the **areal radii**.

There are two ways to derive the standard redshift formula from (5).

i) *Eclectic approach.* The condition

$$(b-a)/(a\eta_a^5) << 1 \tag{6}$$

implies the validity of the geometric optics approximation. Thence one can write $\hat{P}_R(R_0) = \hbar\hat{\omega}_{R_0}N_{R_0}$ and $\hat{P}_R(R) = \hbar\hat{\omega}_R N_R$, where $\hat{\omega}_{R_0}$ and $\hat{\omega}_R$ are the mean frequencies of the initial and final pulses (as measured by static observers) and N_{R_0} and N_R are the respective photon fluxes. If the photon fluxes are conserved, then from (5) one infers

$$\hat{\omega}_R = \sqrt{\frac{\eta_{R_0}}{\eta_R}}\hat{\omega}_{R_0}. \tag{7}$$

ii) *Classical approach.* One can Fourier-analyze the quantity representing the strength field tensor, $h = (-\partial_t + \partial_{r^*})\Psi$; this is preserved along $\tilde{\Gamma}_{R_0}$, if condition (6) is satisfied. Let the spectral strength field density be $g(\omega_\tau) = \int dt e^{-i\omega_\tau t}(-\partial_t + \partial_{r^*})\Psi$. Let the support of $g(\omega)$ be $\tilde{\Omega}$. If one assumes the normalization condition $\int_{\tilde{\Omega}} d\omega_\tau |g(\omega_\tau)| = 1$, then the average frequency can be defined as $\hat{\omega}_\tau = \int_{\tilde{\Omega}} d\omega_\tau \omega_\tau |g(\omega_\tau)|$. When the analysis is done with respect the asymptotic time t, then asymptotic mean frequency $\hat{\omega}$ is found. The Fourier analysis with respect the proper time $d\tau = \sqrt{\eta_R}dt$ of a static observer located at R gives a corresponding mean frequency $\hat{\omega}/\sqrt{\eta_R}$. Thus the two frequencies satisfy (7).

III. COUNTEREXAMPLE TO THE STANDARD REDSHIFT FORMULA

It was pointed out in [5] that (7) is valid for very compact initial pulses - those satisfying the assumption (6). This condition demands not only that a pulse is compact, but also that its relative width is small in comparison to the relative distance from the black hole horizon. It implies that the approximation of the geometric optics is valid and that the backscatter is absent. We conjecture, basing on a numerical evidence, that this can be relaxed to $(r^*(b) - r^*(a))/(2m) << 1$.

In the case of a radiative pulse that is exposed to a significant backscattering the relation (7) would not hold. Let us again review arguments in favour of this conclusion. A wave pulse is superposed from monochromatic waves of different frequencies. When resolved spectrally by an observer, the pulse can be seen as a collection of peaks, each characterized by some mean frequency. The effect of the backscattering is more pronounced for low frequency waves than for high frequency ones - the spectral amplitudes of the former are stronger damped than of the latter. Therefore one would expect that mean frequencies $\hat{\omega}_{R_0}, \hat{\omega}_R$ of the initial and final pulses will not conform to (7). The observed mean frequency $\hat{\omega}_R$ of an outgoing pulse of radiation can be blue-shifted in comparison to the value $\sqrt{\frac{\eta_{R_0}}{\eta_R}}\hat{\omega}_{R_0}$. In these circumstances an attempt to fit observed data of a resolved multi-peak spectrum to the simple scaling law of (7) can lead to redshifts depending on the frequency.

Our aim is to find numerical solutions corresponding to compact initial data, satisfying the compactness assumption $b - a << a$, such that:

i) the spectral amplitude as a function of ω has several peaks with well resolved mean frequencies;

ii) the evolution exhibits significant backscatter (hence the energy flux is diminished);

iii) some of mean frequencies at R_0 and R do not comply to (7).

The condition $b - a << a$ (we remind the reader that a and b are areal radii) is imposed in order to guarantee that initially the wavelengths are much smaller than the radius of the wave front and the curvature radius, i.e., that the assumption of the geometric optics is satisfied at the emission region. On the other hand we **do not assume** the validity of (6); our initial data are such that $r^*(b) - r^*(a) = 4\pi m$, which breaks down this inequality. Let us remark that with such data one can have the geometric optics approximation being valid both at the emission and detection regions, but being evidently broken around $R \approx 3m$, when the wavelengths and the curvature radius are of the same order.

Below we present data describing one of many analysed examples. The mass is normalized to unity and the Schwarzschild radius R_g is equal to 2. Let $I \equiv (r^*(a), r^*(b))$, the support of initial radiation, be comprised between $a = 2 + 1.97 \times 10^{-18}$ and $b = 2 + 10^{-15}$ or, in terms of the tortoise coordinate, between $r^*(b) \approx -68.46$ and $r^*(a) \approx -68.46 - 4\pi$. The purely outgoing initial data have the form $\tilde{\Psi} = \partial_t f(r^* - t) + f(r^* - t)/R$ [7], where f is a free datum in I but vanishes to the left from a. With I being so compact in terms of the areal radius it is reasonable to ignore the R - dependence in this expression and to prescribe initial Ψ as being dependent only on $r^* - t$. Thus the initial data can be defined by prescribing $\tilde{\Psi}$ instead of f. We choose $\Psi(r^*) = \sin^2((r^* - r^*(a)))$ and $\partial_t\Psi(r^*) = -\sin(2(r^* - r^*(a)))$ for $r^* \in I \equiv (r^*(a), r^*(b))$ and $\Psi(r^*) = \partial_t\Psi = 0$ outside I.

Notice that the size of the support of the initial pulse, $i = 4\pi$, satisfies the inequality $i > 2$, which is the plausibility condition (according to the foregoing conjecture) for a significant backscatter.

The signal is measured during the time interval 32π at two observation points, $r_0^* = -68.44$ and $r_1^* = 199.16$;

in terms of the areal radius we have $R_0 = 2 + 1.012 \times 10^{-15}$ and $R = 190.07$. The grid was 120000×60000. Two snapshots of the evolving configuration taken at the above pair of observation points are presented in Fig. 1.

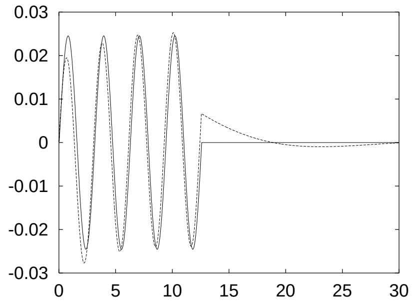

FIG. 1. Temporal dependence of $h \equiv (-\partial_t + \partial_{r^*})\Psi$ (with values of h put on the ordinate) as observed at $R_0 = (2 + 10^{-15})m$ (solid line) and at $R = 190.07m$ (broken line). The time (with values put on the abscissa) is measured from the moment of the arrival of the wave front at each of the observation points. The part of the radiation seen at R (broken line) after $t = 4\pi m$ consists solely of the backscattered radiation. The signal is negligible after $t > 30m$ - its amplitude is much smaller than that of the oscillatory part.

The main pulse passes through R in less than 4π (in units of m) while the scattered wave appears later and quickly vanishes, becoming negligible after $t = 30m$, when the tail dominates. The decay of the tail asymptotics should be like t^{-5}, according to [10]. This asymptotic exponent was in fact obtained in many trial runs of our numerical code with initial pulses located outside $a = 3m$ [11].

Figure 2 in turn shows the spectral decomposition of $h = (-\partial_t + \partial_{r^*})\Psi$, obtained by employing the FOURCO package (netlib). Fast Fourier Transform (IMSL library) was also used, with the purpose of checking the numerical results coming from the application of FOURCO. We depicture **normalized frequencies** in order to see clearly the frequency shift that is caused by the backscatter. The spectrum that is seen at R_0 (solid line) is rescaled by $\sqrt{\eta_{R_0}}$ and the spectrum seen at R (broken line) is rescaled by the factor $\sqrt{\eta_R}$. In the case of negligible backscattering both rescaled spectra would have to coincide.

Figure 2 shows the spectral amplitudes $|g(\omega)|$ in function of the frequency ω. There is a number of peaks, seen at both observation points. We calculated average frequency for each of the resolved peaks, by choosing as the support $\tilde{\Omega}$ of each pulse the interval between the consecutive minima of $|g(\omega)|$. In the case of the lowest frequency peak one can observe a 25-percent blueshift, from 0.25 to 0.31, with variances 0.1 and 0.08, respectively; the broken and solid lines of Fig. 2 coincide for $\omega > 0.5/m$. In the remaining peaks the effect is seemingly absent. This can be interpreted as demonstrating that this backscatter-induced blueshift is frequency dependent. $1/(4m)$ is approximately the asymptotic frequency of the quasinormal modes. It can be seen from Fig. 2 that the effect of the backscattering strongly weakens modes having frequencies ω_b (as observed by an observer static at b), such that $\omega_a\sqrt{\eta_b}$ (which would be the frequency detected by an asymptotic static observer) is smaller than $1/(8m)$ - a half of the asymptotic frequency of quasinormal modes.

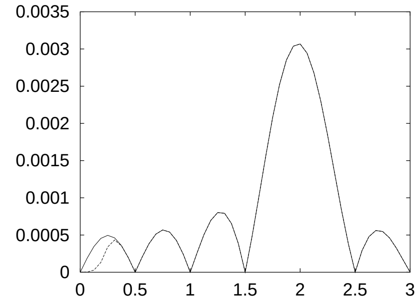

FIG. 2. Frequency amplitude $|g(\omega)|$ (depictured on the ordinate) as observed at $R_0 = (2 + 10^{-15})m$ (solid line) and at $R = 190.07m$ (broken line). The **normalized frequencies** (see the text above) are put on the x-axis (scaled in units of $1/m$).

Modes with $\omega_b\sqrt{\eta_b} > 1/(2m)$ are unchanged, while those in the interval $(1/(8m) < \omega_b\sqrt{\eta_b} < 1/(2m)$ undergo a damping of the spectral amplitude, which is the stronger the smaller the frequency. The physical reason for this is following. Our initial wave pulse has to penetrate through the potential barrier having maximum at $r = 3m$; the modes with frequencies close to those of quasinormal modes are much stronger scattered and their transmission coefficient is small.

Let us point out that the backscatter is negligible – due to the weakness of the effective potential – as long as a wave pulse remains in a region characterized by the tortoise variable $r^* << -1$. Therefore the standard redshift formula applies in the zone $r^* << 1$. Define a shifted radiation pulse as follows: $\Psi(r^* + D) = \Psi(r^*)$ for $D < 0$). Notice that this translated pulse will possess a **different** frequency profile. Let b_D be defined by the equation $r^*(b) + D = b_D + 2\ln(b_D/(2m) - 1)$ and let (w_b) be a frequency measured by a static observer located at $r^*(b)$. A static observer located at $r^*(b) + D$ will measure a blueshifted frequency $(w_b)\sqrt{\eta_b/\eta_{b_D}}$. On the other hand **a static observer located at infinity would detect the same set of frequencies irrespective of** D. Moreover, this observer would notice that the deviation from the redshift formula (if that can be seen in the detected frequency range) does not depend on the position of the initial pulse.

If however a **physical** source of a wave is fixed and shifted from $r^*(b) << -1$ by some $D < 0$ then the two local sets of frequencies $(\omega_{bi}, \omega_{b_D i})$ measured by two (shifted by a distance D) local static observers **do coincide**, $\omega_{bi} = \omega_{b_D i}$. (We assume that tidal effects can be ignored.) The static observer at infinity would then see: i) two **different sets of redshifted frequencies**, $(\omega_{bi,\infty})$ and $(\omega_{b_D i,\infty})$, with $\omega_{bi,\infty} = \sqrt{\eta_b}\omega_{bi}$ and $\omega_{b_D i,\infty} = \sqrt{\eta_{b_D}}\omega_{b_D i}$ for all frequencies which satisfy conditions $\omega_{bi,\infty} >> 1/(4m)$ and $\omega_{b_D i,\infty} >> 1/(4m)$; ii) those frequencies, for which $\omega_{bi,\infty} \leq 1/(4m)$ and/or $\omega_{b_D i,\infty} \leq 1/(4m)$, would disobey the standard redshift formula. It is clear that for any fixed wave source with frequencies ω_i one can find a location b close to the black hole horizon such that some of (the would-be asymptotically detected) frequencies $\eta_b\omega_i$ are of the order of $1/(4m)$, so that (according to our numerical example) one could see a failure of the standard redshift formula.

If that numerically discovered phenomen is generic, then there would exist a natural cutoff (of the order of $1/(4m)$) for the asymptotically observed frequency of those wave pulses that are close to an event horizon. A wave of a mean frequency ω_b satisfying the condition $\omega_b\sqrt{\eta_a} << 1/(4m)$ would not be observed by a distant observer. In the light of the above the heuristic analysis that was reported in the beginning of this section, should be probably supplemented by adding the following: *if a wave peak possesses a contribution with frequencies much smaller than* $1/(4m)$ *then (and only then) its mean frequency will be influenced by the backscatter.*

IV. FINAL REMARKS

The backscattering can modify the relation between initial and asymptotic frequencies. A characteristic frequency (circa $1/(4m)$ in the hitherto used units) that appears useful in this context is (in standard units) $f_c \approx 0.25 \times 31484 \times M_0/m$ Hz, where M_0 is the solar mass. Let us remark that f_c is the frequency typical for the quasinormal modes of the electromagnetic radiation propagating in the Schwarzschild spacetime.

A numerical example of Sec. III reveals a robust difference between the standard redshift prediction (7) and the actual observation. It is observed that for a radiation pulse having an asymptotic frequency $f_\infty \approx f_c$ the initial frequency f_R is noticeably smaller than $f_c/\sqrt{\eta_R}$, a number following from the redshift formula (7)). The peaks with frequencies $f_\infty >> f_c$ are seemingly not influenced by the backscattering. This probably is not always true for all radiation pulses having mean frequencies much bigger than f_c, but we believe that in the latter case the deviation from the redshift formula must be small. Rephrasing this fact in terms of initial data, it is probably fair to say that $(r^*(b) - r^*(a) << 2m$ is the sufficient condition for the validity of the redshift formula (7). That is a much less stringent condition than that used in [5]. Let us stress that in the case of astrophysical black holes of stellar origin the deviation from the law (7) can be seen by an asymptotic observer in the radio part of the electromagnetic spectrum, with wavelengths of the order of 100 km; we do not claim that this phenomenon is of astrophysical interest.

Let us point out that if the backscattering is negligible then: i) all of the energy of an outgoing pulse would get to infinity; ii) its energy flux would be diminished, according to (4). The only mechanism that can imply the loss of the radiation energy is through the backscatter of the radiation off the curvature of the spacetime. This is clearly shown in the wave formulation [7], but a consistent quantum field theoretic treatment (in the curved Schwarzschild spacetime) should give the same answer.

In the present paper the consideration is focused only on the dipole term, but a similar analysis with the same conclusions can to be done in any multipole order. An analogous phenomenon, leading to the demodulation of gravitational signals, will also manifest in the spectra of gravitational waves; this in principle may be detected.

Acknowledgments. This work has been suported in part by the KBN grant 2 PO3B 006 23.

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[11] The integration time, that would be needed in order to obtain the asymptotic tail behaviour in the case of the initial data of Sec. III, exceeds by far the integration time required in order to solve our task.